On stability of left-invariant totally geodesic unit vector fields on three-dimensional Lie groups.

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Abstract

We consider the problem on stability or instability of unit vector fields on three-dimensional Lie groups with left-invariant metric which have totally geodesic image in the unit tangent bundle with the Sasaki metric with respect to classical variations of volume. We prove that among non-flat groups only $SO(3)$ of constant curvature $+1$ admits stable totally geodesic submanifolds of this kind. Restricting the variations to left-invariant (i.e., equidistant) ones, we give a complete list of groups which admit stable/unstable unit vector fields with totally geodesic image.

Introduction

Let (M, g) be the Riemannian manifold and ξ be a unit tangent vector field on M. Then ξ can be considered as a local or global (if exists) immersion $\xi : M \to T_1M$ into the unit tangent bundle. The Sasaki metric $\tilde{g}$ on TM gives rise to the metric on T_1M and hence on $\xi(M)$. In this way $(\xi(M), \tilde{g})$ gets definite intrinsic and extrinsic geometry. Particularly, a unit vector field is said to be minimal or totally geodesic if $\xi(M)$ is a minimal or totally geodesic submanifold in $(T_1M, \tilde{g})$. From the variation theory viewpoint, a minimal unit vector field is a stationary point of the first local normal variation of the volume functional of $\xi(M)$. In other words, ξ is *minimal unit vector field* if the mean curvature vector of $\xi(M) \subset (T_1M, \tilde{g})$ vanishes; ξ is *totally geodesic unit vector field* if all the second fundamental forms of $\xi(M) \subset (T_1M, \tilde{g})$ vanish. We refer to this kind of minimality as *the classical*.

A different type of volume variations and hence the minimality for a given unit vector field was proposed in [12] and developed in [9, 10]. Denote by $\mathfrak{X}^1(M)$ a space of all smooth unit vector fields on M. The variation of ξ within $\mathfrak{X}^1(M)$ gives rise to variation of $\xi(M)$ and hence the volume functional $Vol_\xi : \mathfrak{X}^1(M) \to \mathbb{R}$. We call this type of variations by *the field variations*. A unit vector field ξ was called *minimal*, if ξ is a stationary point of the latter functional. It was proved that this definition of minimality is equivalent to the classical one, i.e. minimal unit vector field gives rise to minimal immersion $\xi : M \to T_1M$. The minimality condition in a meaning of [10] was expressed in terms of a special 1-form. There was constructed a number of examples of minimal unit vector fields by using this 1-form [3, 4, 5, 13, 14, 8] (the list is not complete). In the case of three-dimensional Lie group G with the left-invariant metric, K. Tsukada and L. Vanhecke manage to find a list of all *minimal* left-invariant unit vector fields [17]. It was proved that each minimal left-invariant unit vector field on three-dimensional unimodular Lie group is an eigenvector of the Ricci operator.

Prof. Borisenko A. (Sumy State University, Ukraine) was the first who asked on unit vector fields with *totally geodesic image* in the unit tangent bundle of Riemannian manifold. The author solved the problem in two-dimensional case [19] and have extracted the subclass of totally geodesic fields on three-dimensional Lie groups by equalizing to zero the whole second fundamental form [21]. As a result, it was proved that each totally geodesic left-invariant unit vector field on three-dimensional unimodular Lie group is the unit eigenvector of the Ricci operator of G with the eigenvalue 2, if exists.

The *second variation formula* for the $\xi(M)$-volume functional with respect to the field variation was obtained in [11] and is very complicated to handle with. That is why only little number of results concerning stability/instability are known. Particularly, a minimal unit vector field on 2-dimensional Riemannian manifold is always stable with respect to the field variations [11]. In application to the Hopf vector field on the unit 3-sphere it was also proved that it is minimal and stable [11]. Remark that the Hopf vector field is totally geodesic one as well as the unit characteristic vector field of the Sasakian structure [18].

On the other hand, there is a well-known formula for the second variation of volume [16] which allows to check stability/instability of minimal submanifold in the Riemannian space with respect to local (or global, if admissible) normal variations of the submanifold. We refer to this kind of stability as *classical*. This kind of stability/instability is different from the one considered in [11] because the normal variation of the $\xi(M)$ gives rise to the wider class of the field variations. Namely, the variation field can be non-orthogonal to ξ.

In some cases the normal variation of the minimal submanifold $\xi(M) \subset T_1M$ is probably equivalent to the field variation of minimal unit vector field. The case of totally geodesic left-invariant unit vector field on the three-dimensional Lie group with the left-invariant metric gives a corresponding example. In [14] the authors tried to check stability/instability of left-invariant unit vector fields from Tsukada-Vanhecke list [17]. They have constructed the left-invariant variations of minimal unit left-invariant vector field on compact quotient of unimodular three-dimensional Lie groups which produce instability with respect to the field variations.

In this paper we check the list of all totally geodesic left-invariant unit vector fields on three-dimensional Lie group G with the left-invariant metric and give stability or instability conditions for them with respect to classical normal variations of domains in $\xi(G) \subset T_1G$. In the case of unimodular groups we conduct a complete proof for their compact quotients for the sake of simplicity.

The main result (Theorem 2.2) says that *only $SO(3)$ of constant curvature $+1$ admits classically stable totally geodesic left-invariant unit vector field.* We also give a list of left invariant *totally geodesic* unit vector fields on unimodular three-dimensional Lie groups with the left-invariant metric which are stable/unstable with respect to classical *left-invariant* variations (Theorem 2.4).

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1 Preliminaries.

The definition of the Sasaki metric is based on the bundle projection differential $\pi_*: TTM \to TM$ and the connection map $\mathcal{K}: TTM \to TM$ [7]. For any $\tilde X, \tilde Y \in T_{(q,\xi)}TM$, we have

$$\tilde g(\tilde X, \tilde Y) = g(\pi_*\tilde X, \pi_*\tilde Y) + g(\mathcal{K}\tilde X, \mathcal{K}\tilde Y).$$

By definition, the *vertical* distribution $\mathcal{V}_{(q,\xi)} = \ker \pi_*$ and the *horizontal* one $\mathcal{H}_{(q,\xi)} = \ker \mathcal{K}$. Then $T_{(q,\xi)}TM = \mathcal{V}_{(q,\xi)} \oplus \mathcal{H}_{(q,\xi)}$ and the horizontal and vertical distributions are mutually orthogonal with respect to $\tilde g$.

The *horizontal and vertical lifts* of a vector field X on the base are defined as the unique vector fields X^h and X^v on TM such that

$$\begin{aligned} &\pi_* X^h = X, \quad \pi_* X^v = 0,\\ &\mathcal{K}X^h = 0, \quad\ \ \mathcal{K}X^v = X. \end{aligned}$$

The h- and v- lifts of the tangent frame on M form the lifted frame on TM. As concerns the unit tangent bundle, the lifted frame on T_1M at $(q,\xi) \in T_1M$ is formed by h- lift and the *tangential lift* [2] of the frame on M. The latter is defined by

$$X^{tan} = X^v - g(X,\xi)\xi^v.$$

Evidently, $X^{tan} = X^v$ for all X from the orthogonal complement of the "vector part" of a point (q,ξ). *We use this fact without special comments.*

Denote by $\mathfrak{X}(M)$ the Lie algebra of smooth vector fields on M and by $\mathfrak{X}_{\xi^\perp}(M)$ the orthogonal complement of a unit vector field ξ in $\mathfrak{X}(M)$. If ξ is a *unit vector field* on M, then it can be considered as a mapping $\xi : M \to T_1M$. Then its differential ξ_* sends a vector field $X \in \mathfrak{X}(M)$ into $T\xi(M)$ by [20]

$$\xi_* X = X^h + (\nabla_X \xi)^{tan} = X^h + (\nabla_X\xi)^v,$$

where ∇ means the Riemannian connection of (M,g).

In what follows we use the notion of the *Nomizu operator* $A_\xi : \mathfrak{X}(M) \to \mathfrak{X}_{\xi^\perp}(M)$ given by

$$A_\xi X = -\nabla_X \xi.$$

Denote by A^t_ξ a conjugate Nomizu operator defined by $g(A_\xi X, Y) = g(X, A^t_\xi Y)$. Then one can define the tangent $\xi_* : \mathfrak{X}(M) \to T\xi(M)$ and the normal $\nu : \mathfrak{X}(M) \to T^\perp\xi(M)$ mappings by

$$\begin{aligned} &\xi_*(X) = X^h - (A_\xi X)^{tan} = X^h - (A_\xi X)^v,\\ &\nu(Y) = (A^t_\xi Y)^h + Y^{tan}. \end{aligned} \tag{1}$$

Then there are local orthonormal frames $(e_1,\dots,e_n) \in \mathfrak{X}(M)$ and $(f_1,\dots,f_{n-1}) \in \mathfrak{X}_{\xi^\perp}$ such that

$$A_\xi e_i = \sigma_i f_i, \quad A^t_\xi f_i = \sigma_i e_i,$$

where $\sigma_i \geq 0$ are *the singular values* of the linear operator A_ξ. In fact, e_i are the eigenvectors of the symmetric linear operator $A^t_\xi A_\xi$ and its eigenvalues are the squares of the singular values.

By dimension reasons, there is at least local unit vector field e_0 such that $A_\xi e_0 = 0$. Then

$$\begin{aligned} \tilde e_\alpha &= \tfrac{\xi_*(e_\alpha)}{|\xi_*(e_\alpha)|} = \tfrac{1}{\sqrt{1+\sigma_\alpha^2}}(e_\alpha^h - \sigma_\alpha f_\alpha^v), \ \tilde e_n = e_0^h, \\ \tilde n_\alpha &= \tfrac{\nu(f_\alpha)}{|\nu(f_\alpha)|} = \tfrac{1}{\sqrt{1+\sigma_\alpha^2}}(\sigma_\alpha e_\alpha^h + f_\alpha^v) \quad \alpha = 1, \dots, n-1 \end{aligned} \tag{2}$$

form the tangent and normal framing over $\xi(M) \subset T_1M$. We call this framing the *singular* one. If ξ is a geodesic unit vector field, i.e. $A_\xi \xi = 0$, then one can always put $\tilde e_n = \xi^h$.

Let $\tilde n = \frac{\nu(Z)}{|\nu(Z)|}$ be a unit normal vector field on $\xi(M)$ and $F \subset M$ be a domain with a compact closeur. Denote by $\tilde N = w\tilde n$ a local normal variation vector field, where $w : F \to \mathbb{R}$ is a smooth function such that $w|_{\partial \bar F} = 0$. Suppose $\xi(M)$ is minimal. Then the formula for second variation of the volume in application to our case takes the form

$$\delta^2(Vol_\xi) = \int\limits_{\xi(F)} \Big(||\tilde\nabla^\perp \tilde N||^2 - (\widetilde{Ric}(\tilde N) + ||\tilde S_{\tilde N}||^2) \Big) dVol_\xi,$$

where $\tilde\nabla^\perp$ means the covariant derivative in the normal bundle of $\xi(M)$, $\widetilde{Ric}(\tilde N)$ is the partial Ricci curvature and $\tilde S$ is the shape operator of $\xi(M)$.

In the case of *compact orientable* M and *totally geodesic* $\xi(M)$ the formula takes a simpler form, namely

$$\delta^2 Vol_\xi = \int\limits_{\xi(M)} \sum_{i=1}^n \Big(||\tilde\nabla^\perp_{\tilde e_i} \tilde N||^2 - w^2 \tilde K(\tilde e_i, \tilde n) \Big)\, dVol_\xi.$$

Finally remark, that

$$dVol_\xi = \sqrt{\det(I + A_\xi^t A_\xi)}\, dV := L^{1/2} dV,$$

where dV is the volume element of the base manifold. That is why one can rewrite the formula of the second variation as follows

$$\delta^2 Vol_\xi = \int\limits_M \sum_{i=1}^n \Big(||\tilde\nabla^\perp_{\tilde e_i} \tilde N||^2 - w^2 \tilde K(\tilde e_i, \tilde n) \Big)\, L^{1/2} dV. \tag{3}$$

In the next sections we simplify this formula in application to three-dimensional Lie groups with the left-invariant metric.

2 Three-dimensional unimodular Lie groups with the left-invariant metric.

Let ξ be a unit left-invariant vector field on the three-dimensional Lie group G with the left-invariant Riemannian metric. The group G is unimodular if and only if there is a discrete subgroup Γ acting on G by left translations free and properly discontinuous such that the left quotient $\Gamma \backslash G$ is compact [15]. The $\Gamma \backslash G$ is a compact Riemannian manifold with the same curvature properties as G. The descended unit vector field has the same properties concerning minimality, harmonicity etc. as the one on G [14].

For each three-dimensional unimodular Lie group G, there is an orthonormal frame e_1, e_2, e_3 such that [15]

$$[e_2, e_3] = \lambda_1 e_1, \quad [e_3, e_1] = \lambda_2 e_2, \quad [e_1, e_2] = \lambda_3 e_3. \tag{4}$$

We will refer to this frame as to the *canonical* one. This frame consists of *eigenvectors* of the Ricci curvature operator. Each frame vector field is the *Killing* one and hence *geodesic*. The *Levi-Civita connection* coefficients on G can be easily find, namely $\mu_i = \frac{1}{2}(\lambda_1+\lambda_2+\lambda_3)-\lambda_i$, and the frame covariant derivatives takes the form $\nabla_{e_i} e_k = \mu_i\, e_i \times e_k$. It is also well-known that the *principal Ricci curvatures* are $\rho_i = 2\mu_j\mu_k$ and *the basic sectional curvatures* are $k_{ij} := g(R(e_i, e_j)e_j, e_i) = \frac{1}{2}(\rho_i + \rho_j - \rho_k)$, where i, j, k are all different.

The constants $\lambda_1, \lambda_2, \lambda_3$ define the topological structure of G in the following sense:

Signs of $\lambda_1, \lambda_2, \lambda_3$	Associated Lie group
$+, +, +$	$SO(3)$
$+, +, -$	$SL(2, \mathbb{R})$
$+, +, 0$	$E(2)$
$+, 0, -$	$E(1, 1)$
$+, 0, 0$	Nil^3 (Heisenberg group)
$0, 0, 0$	$\mathbb{R} \oplus \mathbb{R} \oplus \mathbb{R}$

The class of left-invariant totally geodesic unit vector fields on three-dimensional unimodular Lie group G can be described as the eigenvectors of the Ricci operator associated with the eigenvalue 2, if exists [21]. Namely,

ρ_1	ρ_2	ρ_3	μ_1	μ_2	μ_3	ξ
0	0	0	0	0	0	$\mathcal{S}$
0	0	0	$\neq 0$	0	0	$\pm e_1$, $\cos t\, e_2 + \sin t\, e_3$
0	0	0	0	$\neq 0$	0	$\pm e_2$, $\cos t\, e_1 + \sin t\, e_3$
0	0	0	0	0	$\neq 0$	$\pm e_3$, $\cos t\, e_1 + \sin t\, e_2$
2						$\pm e_1$
	2					$\pm e_2$
		2				$\pm e_3$
2	2					$\cos t\, e_1 + \sin t\, e_2$
2		2				$\cos t\, e_1 + \sin t\, e_3$
	2	2				$\cos t\, e_2 + \sin t\, e_3$
2	2	2				$\mathcal{S}$

where $\mathcal{S}$ stands for vector fields of the form $\xi = \cos t \cos s\, e_1 + \cos t \sin s\, e_2 + \sin t\, e_3$ with arbitrary fixed parameters t and s . The detailed analysis of the table above yields the following result [21].

Theorem 2.1 *Let G be three-dimensional unimodular Lie group with the left-invariant metric. Let $\{e_i,\ i = 1,2,3\}$ be the canonical frame of its Lie algebra. Set for definiteness $\lambda_1 \ge \lambda_2 \ge \lambda_3$. Then the totally geodesic left-invariant unit vector fields on (compact quotient of) G are the following:*

G or $\Gamma\backslash G$	*Conditions on $\lambda_1, \lambda_2, \lambda_3$*	ξ
SO(3)	$\lambda_1 = \lambda_2 = \lambda_3 = 2$	$\mathcal{S}$
	$\lambda_1 = \lambda_2 = \lambda > \lambda_3 = 2$	$\pm e_3$
	$\lambda_1 = \lambda_2 = \lambda > 2 > \lambda_3 = \lambda - \sqrt{\lambda^2 - 4}$	$\cos t\, e_1 + \sin t\, e_2$
	$\lambda_1 = 2 > \lambda_2 = \lambda_3 = \lambda > 0$	$\pm e_1$
	$\lambda_1 = \lambda + \sqrt{\lambda^2 - 4} > \lambda = \lambda_2 = \lambda_3 > 2$	$\cos t\, e_2 + \sin t\, e_3$
	$\lambda_1 > \lambda_2 > \lambda_3 > 0,\quad \lambda_m^2 - (\lambda_i - \lambda_k)^2 = 4$	$\pm e_m$ *(i,k,m=1,2,3)*
$SL(2,R)$	$\lambda_3^2 - (\lambda_1 - \lambda_2)^2 = 4$	$\pm e_3$
	$\lambda_1^2 - (\lambda_2 - \lambda_3)^2 = 4$	$\pm e_1$
$E(2)$	$\lambda_1 = \lambda_2 > 0,\quad \lambda_3 = 0$	$\pm e_3,\ \cos t\, e_1 + \sin t\, e_2$
	$\lambda_1^2 - \lambda_2^2 = 4,\ \lambda_1 > \lambda_2 > 0,\quad \lambda_3 = 0$	$\pm e_1$
$E(1,1)$	$\lambda_3^2 - \lambda_1^2 = 4,\ \lambda_1 > 0, \lambda_2 = 0,\quad \lambda_3 < 0$	$\pm e_3$
	$\lambda_1^2 - \lambda_3^2 = 4,\ \lambda_1 > 0, \lambda_2 = 0,\quad \lambda_3 < 0$	$\pm e_1$
Nil^3	$\lambda_1 = 2,\quad \lambda_2 = 0, \lambda_3 = 0$	$\pm e_1$
$R \oplus R \oplus R$	$\lambda_1 = \lambda_2 = \lambda_3 = 0$	$\mathcal{S}$

where $\mathcal{S}$ stands for vector fields of the form $\xi = \cos t \cos s\, e_1 + \cos t \sin s\, e_2 + \sin t\, e_3$ with arbitrary fixed parameters t and s.

For any left invariant vector field $\xi = x_1 e_1 + x_2 e_2 + x_3 e_3$ we have $\nabla_{e_i}\xi = \mu_i\, e_i \times \xi$ and as a consequence, with respect to the canonical frame, we have

$$A_\xi = \begin{pmatrix} 0 & -\mu_2 x_3 & \mu_3 x_2 \\ \mu_1 x_3 & 0 & -\mu_3 x_1 \\ -\mu_1 x_2 & \mu_2 x_1 & 0 \end{pmatrix} \tag{5}$$

To calculate the integrand in (3), we need some Lemmas.

Lemma 2.1 *Let $\xi := e_m$ be a totally geodesic left-invariant unit vector field on (compact quotient of) unimodular three-dimensional Lie group G with left-invariant metric. Then the normal bundle connection coefficients of $\xi(G)$ with respect to framing* (1) *are*

$$\tilde\gamma^i_{j|s} = -\frac{1}{2} k_{ij}\delta_{sm} \quad ((i<j) \ne m),$$

where k_{ij} means the sectional curvature of G along $e_i \wedge e_j$.

Proof. We will conduct the proof for $\xi = e_3$. Observe that since ξ is supposed totally geodesic, the principal Ricci curvature $\rho_3 = 2$ and hence $\mu_1\mu_2 = \frac{1}{2}\rho_3 = 1$. From (5) we get

$$A_\xi = \begin{pmatrix} 0 & -\mu_2 & 0 \\ \mu_1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad A_\xi^t = \begin{pmatrix} 0 & \mu_1 & 0 \\ -\mu_2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad A_\xi^t A_\xi = \begin{pmatrix} \mu_1^2 & 0 & 0 \\ 0 & \mu_2^2 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Hence, the e_1, e_2 can be taken as the vectors of singular frame. Since

$$A_\xi e_1 = \mu_1 e_2, \quad A_\xi e_2 = -\mu_2 e_1,$$

we may put $\sigma_1 = \mu_1,\ \sigma_2 = \mu_2$ and take $f_1 = e_2, f_2 = -e_1$. Then the framing (2) takes the form

$$\tilde{e}_1 = \Big(\frac{1}{\sqrt{1+\mu_1^2}}e_1\Big)^h - \Big(\frac{\mu_1}{\sqrt{1+\mu_1^2}}e_2\Big)^v, \tilde{e}_2 = \Big(\frac{1}{\sqrt{1+\mu_2^2}}e_2\Big)^h + \Big(\frac{\mu_2}{\sqrt{1+\mu_2^2}}e_1\Big)^v, \tilde{e}_3 = (e_3)^h \tag{6}$$

$$\tilde{n}_1 = \Big(\frac{\mu_1}{\sqrt{1+\mu_1^2}}e_1\Big)^h + \Big(\frac{1}{\sqrt{1+\mu_1^2}}e_2\Big)^v, \quad \tilde{n}_2 = \Big(\frac{\mu_2}{\sqrt{1+\mu_2^2}}e_2\Big)^h - \Big(\frac{1}{\sqrt{1+\mu_2^2}}e_1\Big)^v. \tag{7}$$

Recall, that $\tilde{\gamma}^i_{j|s} := \tilde{g}(\tilde{\nabla}_{\tilde{e}_s}\tilde{n}_j, \tilde{n}_i)$ and in our case we only need to calculate $\tilde{\gamma}^2_{1|s}$. To do this we use Kowalski-type formulas from [2], namely

$$\tilde{\nabla}_{X^h}Y^h = (\nabla_X Y)^h - \tfrac{1}{2}(R(X,Y)\xi)^{tan}, \quad \tilde{\nabla}_{X^h}Y^{tan} = (\nabla_X Y)^{tan} + \tfrac{1}{2}(R(\xi,Y)X)^h,$$

$$\tilde{\nabla}_{X^{tan}}Y^h = \tfrac{1}{2}(R(\xi,X)Y)^h, \quad \tilde{\nabla}_{X^{tan}}Y^{tan} = -g(Y,\xi)X^{tan}.$$

Then

$$\tilde{\nabla}_{X_1^h + X_2^{tan}}(Y_1^h + Y_2^{tan}) =$$
$$\Big(\nabla_{X_1}Y_1 + \frac{1}{2}R(\xi,Y_2)X_1 + \frac{1}{2}R(\xi,X_2)Y_1\Big)^h + \Big(\nabla_{X_1}Y_2 - \frac{1}{2}R(X_1,Y_1)\xi - g(Y_2,\xi)X_2\Big)^{tan}$$

Straight forward calculations show that the curvature tensor components are of the form

	e_1	e_2	e_3
$R(e_1,e_2)\bullet$	$-k_{12}e_2$	$k_{12}e_1$	0
$R(e_1,e_3)\bullet$	$-k_{13}e_3$	0	$k_{13}e_1$
$R(e_2,e_3)\bullet$	0	$-k_{23}e_3$	$k_{23}e_2$

where $k_{ij} = \frac{1}{2}(\rho_i + \rho_j - \rho_m)$ $(i \neq j \neq m \neq i)$ are basic sectional curvatures.

Using this formulas, we obtain easily $\tilde{\nabla}_{\tilde{e}_1}\tilde{n}_1 = ((*)e_3)^{tan} = 0$, $\tilde{\nabla}_{\tilde{e}_2}\tilde{n}_1 = (*)e_3^h$ and hence $\tilde{\gamma}^2_{1|1} = \tilde{\gamma}^2_{1|2} = 0$. Finaly,

$$\tilde{\nabla}_{\tilde{e}_3}\tilde{n}_1 = \frac{1}{2}\frac{k_{12}}{\sqrt{1+\mu_1^2}}e_2^h - \frac{1}{2}\frac{2\mu_3 - \mu_1 k_{13}}{\sqrt{1+\mu_1^2}}e_1^v$$

As $\mu_1\mu_2 = 1$, we can simplify

$$\frac{2\mu_3 - \mu_1 k_{13}}{\sqrt{1+\mu_1^2}} = \frac{2\mu_3 - \mu_1(\mu_1\mu_2 + \mu_2\mu_3 - \mu_1\mu_3)}{\sqrt{1+\mu_1^2}} = \frac{2\mu_3 - (\mu_1 + \mu_3 - \mu_1^2\mu_3)}{\sqrt{1+\mu_1^2}} =$$
$$\frac{\mu_3 + \mu_1^2\mu_3 - \mu_1}{\sqrt{1+\mu_1^2}} = \frac{\mu_2\mu_3 + \mu_1\mu_3 - \mu_2\mu_3}{\mu_2\sqrt{1+\mu_1^2}} = \frac{k_{12}}{\sqrt{1+\mu_2^2}},$$
$$\frac{k_{12}}{\sqrt{1+\mu_1^2}} = \frac{\mu_2 k_{12}}{\sqrt{1+\mu_2^2}}.$$

So, we have

$$\tilde{\nabla}_{\tilde{e}_3}\tilde{n}_1 = \frac{1}{2}k_{12}\tilde{n}_2$$

and hence $\tilde{\gamma}^2_{1|3} = \frac{1}{2}k_{12}$. For the cases of $\xi = e_1$ and $\xi = e_2$ the calculations are similar. ∎

The partial Ricci curvature $\widetilde{Ric}(\tilde{N}) = w^2 \sum_{i=1}^{n} \tilde{K}(\tilde{e}_i, \tilde{n})$, where $\tilde{e}_i$ are the vectors of orthonormal frame tangent to $\xi(M)$, can be calculated by using the formula for the sectional curvature of T_1M. Namely, if $\tilde{X} = X_1^h + X_2^{tan}$ and $\tilde{Y} = Y_1^h + Y_2^{tan}$ are orthonormal, then [6]:

$$\tilde{K}(\tilde{X},\tilde{Y}) = \langle R(X_1,Y_1)Y_1, X_1\rangle - \frac{3}{4}\|R(X_1,Y_1)\xi\|^2 +$$
$$\frac{1}{4}\|R(\xi,Y_2')X_1 + R(\xi,X_2')Y_1\|^2 + 3\langle R(X_1,Y_1)Y_2', X_2'\rangle - \langle R(\xi,X_2')X_1, R(\xi,Y_2')Y_1\rangle +$$
$$\|X_2'\|^2\|Y_2'\|^2 - \langle X_2', Y_2'\rangle^2 + \langle(\nabla_{X_1}R)(\xi,Y_2')Y_1, X_1\rangle + \langle(\nabla_{Y_1}R)(\xi,X_2')X_1, Y_1\rangle, \quad (8)$$

where $X_2' = X_2 - g(X_2,\xi)\xi$, $Y_2' = Y_2 - g(Y_2,\xi)\xi$, R and ∇ are the curvature tensor and Riemannian connection of the base manifold (M,g) respectively.

So, to find the partial Ricci curvature of $\xi(G)$, we need the covariant derivatives of the curvature tensor. One can find them by standard calculations.

Lemma 2.2 *Let (e_1,e_2,e_3) be the canonical left-invariant frame on (compact quotient of) three-dimensional unimodular Lie group with the left-invariant metric. Then the covariant derivatives of the curvature tensor are of he form*

	$(\nabla_\bullet R)(e_1,e_2)e_1$	$(\nabla_\bullet R)(e_1,e_2)e_2$	$(\nabla_\bullet R)(e_1,e_2)e_3$
e_1	$\mu_1(\rho_3-\rho_2)e_3$	0	$-\mu_1(\rho_3-\rho_2)e_1$
e_2	0	$\mu_2(\rho_3-\rho_1)e_3$	$-\mu_2(\rho_3-\rho_1)e_2$
e_3	0	0	0

	$(\nabla_\bullet R)(e_1,e_3)e_1$	$(\nabla_\bullet R)(e_1,e_3)e_2$	$(\nabla_\bullet R)(e_1,e_3)e_3$
e_1	$\mu_1(\rho_3-\rho_2)e_2$	$-\mu_1(\rho_3-\rho_2)e_1$	0
e_2	0	0	0
e_3	0	$\mu_3(\rho_2-\rho_1)e_3$	$-\mu_3(\rho_2-\rho_1)e_2$

	$(\nabla_\bullet R)(e_2,e_3)e_1$	$(\nabla_\bullet R)(e_2,e_3)e_2$	$(\nabla_\bullet R)(e_2,e_3)e_3$
e_1	0	0	0
e_2	$\mu_2(\rho_3-\rho_1)e_2$	$-\mu_2(\rho_3-\rho_1)e_1$	0
e_3	$\mu_3(\rho_2-\rho_1)e_3$	0	$-\mu_3(\rho_2-\rho_1)e_1$

where ρ_i are the principal Ricci curvatures and μ_i are the connection coefficients.

Now we can calculate the partial Ricci curvature with respect to arbitrary normal vector field for totally geodesic $\xi(G)$.

Lemma 2.3 *Let $\xi = e_m$ be a totally geodesic unit vector field on (compact quotient of) three-dimensional unimodular Lie group G with the left-invariant metric. The partial Ricci curvature of $\xi(G)$ with respect to arbitrary normal vector field $\tilde N = h_i\tilde n_i + h_j\tilde n_j$ $(i \neq j \neq m \neq i)$ is of the form*

$$\widetilde{Ric}(\tilde N) = \frac{1}{4}k_{ij}(h_i^2 + h_j^2) + \Big(1 - \frac{\rho_j^2}{4}\Big)h_i^2 + \Big(1 - \frac{\rho_i^2}{4}\Big)h_j^2$$

where k_{ij} is a basic $e_i \wedge e_j$ sectional curvature and ρ_i are the principal Ricci curvatures.

Proof. We will conduct the proof for $\xi = e_3$, since the other cases are similar. Take the $\xi(G)$ tangent and normal framing according to (6) and (7). Then the arbitrary normal vector field $\tilde N$ can be expressed by

$$\tilde N = \left(\frac{h_1\mu_1}{\sqrt{1+\mu_1^2}}e_1 + \frac{h_2\mu_2}{\sqrt{1+\mu_2^2}}e_2\right)^h + \left(-\frac{h_2}{\sqrt{1+\mu_2^2}}e_1 + \frac{h_1}{\sqrt{1+\mu_1^2}}e_2\right)^v$$

Observe that if $\tilde X$ is of unit length and orthogonal to $\tilde Y$, then $|\tilde Y|^2\tilde K(\tilde X, \tilde Y)$ could be calculated by (8) assuming that Y_1 and Y_2 are the components of the non-normalized vector. Keeping this, put

$$Y_1 = \frac{h_1\mu_1}{\sqrt{1+\mu_1^2}}e_1 + \frac{h_2\mu_2}{\sqrt{1+\mu_2^2}}e_2, \quad Y_2 = -\frac{h_2}{\sqrt{1+\mu_2^2}}e_1 + \frac{h_1}{\sqrt{1+\mu_1^2}}e_2.$$

To calculate $\tilde K(\tilde e_1, \tilde N)$, put

$$X_1 = \frac{1}{\sqrt{1+\mu_1^2}}e_1 \qquad X_2 = \frac{-\mu_1}{\sqrt{1+\mu_1^2}}e_2.$$

The calculations with MAPLE yield:

$$\begin{aligned}
&\langle R(X_1,Y_1)Y_1,X_1\rangle = \frac{\mu_2^2k_{12}}{(1+\mu_1^2)(1+\mu_2^2)}h_2^2\Big|_{\mu_1\mu_2=1} = \frac{\mu_1^2\mu_3+\mu_3-\mu_1}{\mu_1(1+\mu_1^2)^2}h_2^2,\\
&||R(X_1,Y_1)\xi||^2 = 0,\\
&||R(\xi,Y_2)X_1 + R(\xi,X_2)Y_1||^2 = \frac{(k_{13}+\mu_1\mu_2k_{23})^2}{(1+\mu_1^2)(1+\mu_2^2)}h_2^2\Big|_{\mu_1\mu_2=1} = \frac{4\mu_1^2}{(1+\mu_1^2)^2}h_2^2,\\
&\langle R(X_1,Y_1)Y_2,X_2\rangle = -\frac{\mu_1\mu_2k_{12}}{(1+\mu_1^2)(1+\mu_2^2)}h_2^2\Big|_{\mu_1\mu_2=1} = -\frac{\mu_1(\mu_1^2\mu_3+\mu_3-\mu_1)}{(1+\mu_1^2)^2}h_2^2,\\
&\langle R(\xi,X_2)X_1, R(\xi,Y_2)Y_1\rangle = 0,\\
&\|X_2\|^2\|Y_2\|^2 - \langle X_2,Y_2\rangle^2 = \frac{\mu_1^2}{(1+\mu_1^2)(1+\mu_2^2)}h_2^2\Big|_{\mu_1\mu_2=1} = \frac{\mu_1^4}{(1+\mu_1^2)^2}h_2^2,\\
&\langle(\nabla_{X_1}R)(\xi,Y_2)Y_1,X_1\rangle = \frac{\rho_3(\rho_2-\rho_3)}{2(1+\mu_1^2)(1+\mu_2^2)}h_2^2\Big|_{\rho_3=2,\mu_1\mu_2=1} = \frac{2\mu_1^2(\mu_1\mu_3-1)}{(1+\mu_1^2)^2}h_2^2,\\
&\langle(\nabla_{Y_1}R)(\xi,X_2)X_1,Y_1\rangle = -\frac{\mu_2^2\rho_3(\rho_1-\rho_3)}{2(1+\mu_1^2)(1+\mu_2^2)}h_2^2\Big|_{\rho_3=2,\mu_1\mu_2=1} = \frac{2(\mu_1-\mu_3)}{\mu_1(1+\mu_1^2)^2}h_2^2.
\end{aligned}$$

After substitution into (8), we get

$$|\tilde{N}|^2 \tilde{K}(\tilde{e}_1, \tilde{N}) = (1 - \frac{1}{2}\rho_1) h_2^2.$$

After similar calculations, one can find

$$\begin{aligned} |\tilde{N}|^2 \tilde{K}(\tilde{e}_2, \tilde{N}) &= (1 - \frac{1}{2}\rho_2) h_1^2, \\ |\tilde{N}|^2 \tilde{K}(\tilde{e}_3, \tilde{N}) &= \frac{1}{4} k_{12}^2 (h_1^2 + h_2^2) + \Big(\frac{\rho_2}{2} - \frac{\rho_2^2}{4}\Big) h_1^2 + \Big(\frac{\rho_1}{2} - \frac{\rho_1^2}{4}\Big) h_2^2. \end{aligned}$$

It follows then

$$\widetilde{Ric}(\tilde{N}) = \frac{1}{4} k_{12}^2 (h_1^2 + h_2^2) + \Big(1 - \frac{\rho_2^2}{4}\Big) h_1^2 + \Big(1 - \frac{\rho_1^2}{4}\Big) h_2^2,$$

which completes the proof.

■

The following Lemma is the principal one.

Lemma 2.4 *Let $\xi = e_m$ be a totaly geodesic left-invariant unit vector field on three-dimensional non-flat compact quotient of unimodular Lie group with the left-invariant metric. Then the integrand in the second volume variation formula* (3) *can be reduced to*

$$W(h,h) := \Big[\frac{e_i(h_i)^2}{1+\mu_i^2} - \frac{2k_{ij}}{\lambda_m} e_i(h_i) e_j(h_j) + \frac{e_j(h_j)^2}{1+\mu_j^2} +$$

$$\frac{e_i(h_j)^2}{1+\mu_i^2} + \frac{2k_{ij}}{\lambda_m} e_i(h_j) e_j(h_i) + \frac{e_j(h_i)^2}{1+\mu_j^2} + e_m(h_i)^2 + e_m(h_j)^2 + \big(\frac{\rho_j^2}{4} - 1\big) h_i^2 + \big(\frac{\rho_i^2}{4} - 1\big) h_j^2\Big] |\lambda_m|, \tag{9}$$

where $i \neq j \neq m \neq i$, ρ_i and ρ_j are the principal Ricci curvatures, k_{ij} are the basic sectional curvatures of G and h_i are the variation functions.

Proof. We will conduct the calculations for the case $m = 3$. First of all observe, that $\xi = e_m$ is the unit Ricci eigenvector of eigenvalue $\rho_3 = 2$, which means that $\mu_1 \mu_2 = 1$ and hence

$$L = \det(I + A_\xi^t A_\xi) = 1 + \mu_1^2 + \mu_2^2 + \mu_1^2 \mu_2^2 = 2 + \mu_1^2 + \mu_2^2 = (\mu_1 + \mu_2)^2 = \lambda_3^2.$$

Therefore, $dVol_\xi$ is a constant multiple of dV, namely $dVol_\xi = |\lambda_3| dV$. Take the $\xi(G)$ tangent and normal framing according to (6) and (7). Put $\tilde{N} = h_1 \tilde{n}_1 + h_2 \tilde{n}_2$. To calculate $|\tilde{\nabla}^\perp_{\tilde{e}_i} \tilde{N}|^2$, observe that

$$\begin{aligned} &\tilde{\nabla}^\perp_{\tilde{e}_i} \tilde{N} = \langle\langle \tilde{\nabla}_{\tilde{e}_i} \tilde{N}, \tilde{n}_1 \rangle\rangle\, \tilde{n}_1 + \langle\langle \tilde{\nabla}_{\tilde{e}_i} \tilde{N}, \tilde{n}_2 \rangle\rangle\, \tilde{n}_2 = \tilde{e}_i(h_1) \tilde{n}_1 + \tilde{e}_i(h_2) \tilde{n}_2 + \\ &h_2 \langle\langle \tilde{\nabla}_{\tilde{e}_i} \tilde{n}_2, \tilde{n}_1 \rangle\rangle\, \tilde{n}_1 + h_1 \langle\langle \tilde{\nabla}_{\tilde{e}_i} \tilde{n}_1, \tilde{n}_2 \rangle\rangle\, \tilde{n}_2 = \tilde{e}_i(h_1) \tilde{n}_1 + \tilde{e}_i(h_2) \tilde{n}_2 + h_2 \tilde{\gamma}^1_{2|i} \tilde{n}_1 + h_1 \tilde{\gamma}^2_{1|i} \tilde{n}_2 \end{aligned}$$

By Lemma 2.1, we have

$$\tilde{\nabla}^\perp_{\tilde{e}_1} \tilde{N} = \tilde{e}_1(h_1) \tilde{n}_1 + \tilde{e}_1(h_2) \tilde{n}_2, \quad \tilde{\nabla}^\perp_{\tilde{e}_2} \tilde{N} = \tilde{e}_2(h_1) \tilde{n}_1 + \tilde{e}_2(h_2) \tilde{n}_2,$$

$$\tilde\nabla^{\perp}_{\tilde e_3}\tilde N = \Big(\tilde e_3(h_1) - \frac{1}{2}k_{12}h_2\Big)\tilde n_1 + \Big(\tilde e_3(h_2) + \frac{1}{2}k_{12}h_1\Big)\tilde n_2.$$

Therefore

$$\sum_{i=1}^{3}||\tilde\nabla^{\perp}_{\tilde e_i}\tilde N||^2 = \sum_{i=1}^{3}\Big(\tilde e_i(h_1)^2 + \tilde e_i(h_2)^2\Big) + k_{12}\big(\tilde e_3(h_2)h_1 - \tilde e_3(h_1)h_2\big) + \frac{1}{4}k_{12}^2(h_1^2+h_2^2).$$

Since h_i are the functions on the base manifold, we have $\tilde e_\alpha(h_\sigma) = \frac{1}{\sqrt{1+\mu_\alpha^2}}e_\alpha(h_\sigma)$ and $\tilde e_3(h_\sigma) = e_3(h_\sigma)$, where $(\alpha,\sigma = 1,2)$. Hence,

$$\sum_{i=1}^{3}||\tilde\nabla^{\perp}_{\tilde e_i}\tilde N||^2 = \sum_{\alpha=1}^{2}\frac{1}{1+\mu_\alpha^2}\Big(e_\alpha(h_1)^2 + e_\alpha(h_2)^2\Big) + \\ e_3(h_1)^2 + e_3(h_2)^2 + k_{12}\big(e_3(h_2)h_1 - e_3(h_1)h_2\big) + \frac{1}{4}k_{12}^2(h_1^2+h_2^2).$$

Since G is compact, by the divergence theorem

$$\int_G div(X)dV = 0$$

for any vector field X. For $X = h_1h_2e_3$, we have

$$div(h_1h_2e_3) = g(\mathrm{grad}(h_1h_2), e_3) = e_3(h_1)h_2 + e_3(h_2)h_1$$

and hence

$$\int_G \Big(e_3(h_2)h_1 - e_3(h_1)h_2\Big)dV = 2\int_G e_3(h_2)h_1dV.$$

Analyzing the Table in the Theorem 2.1 one can observe, that all cases (except $E(2)$ and T^3 with flat metric) the totaly geodesic e_i corresponds to $\lambda_i \neq 0$. Therefore, we can continue as

$$2\int_{G/\Gamma} e_3(h_2)h_1dV = \frac{2}{\lambda_3}\int_G [e_1,e_2](h_2)h_1dV.$$

Expand

$$h_1[e_1,e_2](h_2) = h_1e_1(e_2(h_2)) - h_1e_2(e_1(h_2)) = \\ e_1(h_1e_2(h_2)) - e_1(h_1)e_2(h_2) - e_2(h_1e_1(h_2)) + e_2(h_1)e_1(h_2).$$

Since G is compact and boundaryless, after applying the Stokes formula we get

$$\int_G \Big(e_3(h_2)h_1 - e_3(h_1)h_2\Big)dV = \frac{2}{\lambda_3}\int_G \Big(e_2(h_1)e_1(h_2) - e_1(h_1)e_2(h_2)\Big)dV.$$

Hence,

$$\int_{\xi(G)}\sum_{i=1}^{3}||\tilde\nabla^{\perp}_{\tilde e_i}\tilde N||^2 dVol_\xi = \int_G \Big(\frac{e_1(h_1)^2}{1+\mu_1^2} - \frac{2k_{12}}{\lambda_3}e_1(h_1)e_2(h_2) + \frac{e_2(h_2)^2}{1+\mu_2^2} + \\ \frac{e_1(h_2)^2}{1+\mu_1^2} + \frac{2k_{12}}{\lambda_3}e_1(h_2)e_2(h_1) + \frac{e_2(h_1)^2}{1+\mu_2^2} + e_3(h_1)^2 + e_3(h_2)^2 + \frac{1}{4}k_{12}^2(h_1^2+h_2^2)\Big)|\lambda_3|dV$$

Taking into account the result of Lemma 2.3, we obtain

$$\delta^2 Vol_\xi = \int\limits_G \Big(\frac{e_1(h_1)^2}{1+\mu_1^2} - \frac{2k_{12}}{\lambda_3} e_1(h_1)e_2(h_2) + \frac{e_2(h_2)^2}{1+\mu_2^2} +$$
$$\frac{e_1(h_2)^2}{1+\mu_1^2} + \frac{2k_{12}}{\lambda_3} e_1(h_2)e_2(h_1) + \frac{e_2(h_1)^2}{1+\mu_2^2} + e_3(h_1)^2 + e_3(h_2)^2 + \big(\frac{\rho_2^2}{4} - 1\big)h_1^2 + \big(\frac{\rho_1^2}{4} - 1\big)h_2^2\Big)|\lambda_3| dV.$$

The other cases can be treated in a similar way.

■

Remark 1 It is worthwhile to mention that if $\mu_1 = \mu_2 = \mu_3 = 1$, then $\rho_1 = \rho_2 = \rho_3 = 2$ and the integrand (9) up to multiple 2 is the same as in [11] obtained for the Hopf vector field on $S^3(1)$. In this case we deal with $SO(3)$ of constant curvature $+1$ which is isometric to $S^3(1)$. The left-invariant unit vector field corresponds the Hopf vector field on $S^3(1)$. So we can conclude that in this case the second variation of volume with respect to the field variation is equal to a half of classical second variation of volume. Therefore, the Hopf vector field is stable with respect to both types of variations. The stability the Hopf vector field with respect to the field variations was proved in [11] and in [18] for the classical treatment.

From Lemma 2.4 we immediately conclude the following.

Theorem 2.2 *Let ξ be left-invariant unit vector field on compact quotient of non-flat three-dimensional unimodular Lie group G with left-invariant metric. Then $\xi(\Gamma\backslash G)$ is stable totally geodesic submanifold in $T_1(\Gamma\backslash G)$ if and only if $G = SO(3)$ of constant curvature $+1$ and ξ is arbitrary left-invariant.*

Proof. Let $\xi = e_m$ be totally geodesic. Then $\rho_m = 2$ and to be left-invariant stable, the other Ricci curvatures must satisfy

$$|\rho_i| \ge 2 \quad \text{or,equivalently,} \quad |\mu_m \mu_j| \ge 1$$

and

$$|\rho_j| \ge 2 \quad \text{or,equivalently,} \quad |\mu_i \mu_m| \ge 1.$$

To be generally stable, both quadratic expressions involving derivatives must be positively semi-definite. The latter condition is equivalent to

$$\frac{k_{ij}^2}{\lambda_m^2} \le \frac{1}{(1+\mu_i^2)(1+\mu_j^2)}.$$

Since $\rho_m = 2\mu_i\mu_j = 2$, we have $(1+\mu_i^2)(1+\mu_j^2) = (2+\mu_i^2+\mu_j^2) = (\mu_i+\mu_j)^2 = \lambda_m^2$. As a result, $|k_{ij}| \le 1$. Observe that $k_{ij} = \mu_i\mu_m + \mu_m\mu_j - 1$ and hence the equality $|k_{ij}| \le 1$ is equivalent to

$$0 \le \mu_m(\mu_i+\mu_j) \le 2 \quad \text{or} \quad 0 \le \frac{\mu_m}{\mu_i}(1+\mu_i^2) \le 2 \quad \text{or} \quad 0 \le \frac{\mu_m}{\mu_i} \le \frac{2}{1+\mu_i^2}.$$

Evidently, all connection coefficients have to be of the same sign. Therefore, the classical stability take place if

$$\mu_m\mu_i \geq 1, \quad \frac{\mu_m}{\mu_i} \geq 1, \quad 0 \leq \frac{\mu_m}{\mu_i} \leq \frac{2}{1+\mu_i^2}.$$

The system is compatible if and only if $\mu_1 = \mu_2 = \mu_3 = \pm 1$. Taking into account the signs of λ_i, we obtain a unique solution $\mu_1 = \mu_2 = \mu_3 = 1$ which means that the base manifold is $SO(3)$ of constant curvature $+1$ and hence ξ is arbitrary left-invariant.

If the system is inconsistent, then the totally geodesic submanifold $\xi(G)$ is unstable. Indeed, if say $\rho_i < 2$, then in the case of compact quotient one can take $h_i = 0$, $h_m = 0$ and $h_j = const \neq 0$ and we get $W(h,h) < 0$ over whole compact quotient. If $\rho_i \geq 2$ and $\rho_j \geq 2$ but $|k_{ij}| > 1$, then both quadratic expressions that involve derivatives of h_i and h_j in $W(h,h)$ are not positively semi-definite. By taking $h_3 = 0$ and h_1, h_2 sufficiently small with derivatives making the quadratic expressions negative, we obtain negative $W(h,h)$ at least over some domain $F \subset \Gamma\backslash G$.

∎

The proof of Lemma 2.4 essentially uses non-flatness of the group. If the group is flat, then the second classical variation of volume for the unit vector field with totally geodesic image is much simpler.

Theorem 2.3 *Let ξ is a unit vector field on compact quotient of flat three-dimensional unimodular Lie group G with the left-invariant metric. Then*

- *if $G = E(2)$ and ξ is a parallel unit vector field on $E(2)$, then $\xi(\Gamma\backslash G)$ is stable totally geodesic submanifold;*
- *if $G = E(2)$ and ξ is in integrable distribution orthogonal to the parallel vector field on $E(2)$, then $\xi(\Gamma\backslash G)$ is unstable totally geodesic submanifold;*
- *if $G = R \oplus R \oplus R$ and is arbitrary left-invariant, then $\xi(T^3)$ is stable totally geodesic submanifold.*

Proof. We have flat $E(2)$ if $\lambda_1 = \lambda_2 = a > \lambda_3 = 0$. In this case $\mu_1 = 0, \mu_2 = 0, \mu_3 = a$ and $\rho_1 = \rho_2 = \rho_3 = 0$. The field $\xi = e_3$ is the field of unit normals of the integrable orthogonal distribution $\xi^\perp$. In this case $\widetilde{Ric}(\tilde N) = 0$ and we have $\xi(G)$ **stable** totally geodesic submanifold in T_1G.

As concerns the field $\xi = \cos te_1 + \sin te_2$, rotating the frame in $e_1 \wedge e_2$ plane we may always put $\xi = e_1$ without loss of generality. Then

$$A_\xi = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -a \\ 0 & 0 & 0 \end{pmatrix}, \quad A_\xi^t = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -a & 0 \end{pmatrix}.$$

The tangent frame consists of

$$\tilde e_1 = e_1^h, \quad \tilde e_2 = e_2^h, \quad \tilde e_3 = \frac{1}{\sqrt{1+a^2}}(e_3^h + a e_2^v).$$

The normal frame on $\xi(G)$ consists of

$$\tilde n_2 = \frac{1}{\sqrt{1+a^2}}(-ae_3^h + e_2^v), \quad \tilde n_3 = e_3^v.$$

For the field of normal variation $\tilde N = h_2\tilde n_2 + h_3\tilde n_3$, we have

$$\widetilde{Ric}(\tilde N) = \frac{a^2}{1+a^2}h_3^2.$$

The normal connection of $\xi(G)$ is flat and hence, by choosing the variation with constant h_1 and h_2, we get

$$\delta^2 Vol_\xi = -\frac{a^2}{1+a^2}h_3^2 Vol(G),$$

which means that $\xi(G)$ is **ustable** totally geodesic submanifold in T_1G.

In the case of $R\oplus R\oplus R$ the compact quotient is flat torus T^3. Each left-invariant field is parallel and therefore, the $\xi(T^3)$ is **stable** totally geodesic submanifold.

■

Considering the field of normal variation of $\xi(G)$ for $\xi = e_3$, namely,

$$\tilde N = \left(\frac{h_1\mu_1}{\sqrt{1+\mu_1^2}}e_1 + \frac{h_2\mu_2}{\sqrt{1+\mu_2^2}}e_2\right)^h + \left(-\frac{h_2}{\sqrt{1+\mu_2^2}}e_1 + \frac{h_1}{\sqrt{1+\mu_1^2}}e_2\right)^v$$

one can observe that this field generates two variations of the field ξ in a meaning of [11], namely

$$Z_1 = \pi_*(\tilde N) = \frac{h_1\mu_1}{\sqrt{1+\mu_1^2}}e_1 + \frac{h_2\mu_2}{\sqrt{1+\mu_2^2}}e_2, \quad Z_2 = \mathcal{K}(\tilde N) = -\frac{h_2}{\sqrt{1+\mu_2^2}}e_1 + \frac{h_1}{\sqrt{1+\mu_1^2}}e_2.$$

If h_1 and h_2 are non-constant, then these variations exclude $\xi(G)$ from the class of submanifolds in T_1G, generated by the left invariant unit vector fields. This fact justifies the following definition.

Definition 2.1 *Let ξ be left-invariant unit vector field on Lie group G with the left-invariant metric. The normal variation vector field $\tilde N$ on $\xi(G)\subset T_1G$ is called left-invariant, if $Z_1 = \pi_*(\tilde N)$ $Z_2 = \mathcal{K}(\tilde N)$ are left-invariant vector fields on G.*

If we restrict the variations to the left-invariant ones, we obtain a wider class of classically stable totally geodesic unit vector fields.

Theorem 2.4 *Let G be three-dimensional unimodular Lie group with the left-invariant metric. Let (e_1, e_2, e_3) is the canonical frame of its Lie algebra. Set for definiteness $\lambda_1 \ge \lambda_2 \ge \lambda_3$. Then stable or unstable with respect to left-invariant variations totally geodesic submanifolds generated by unit left-invariant vector field ξ on compact quotient of G are the following.*

G or $\Gamma\backslash G$	*Ricci principal curvatures*	ξ	*left-invariant stability or instability*
SO(3)	$\rho_1 = \rho_2 = \rho_3 = 2$	$\mathcal{S}$	*stable*
	$\rho_1 = \rho_2 > \rho_3 = 2$	$\pm e_3$	*stable*
	$\rho_1 = \rho_2 = 2 > \rho_3$	$\cos t\, e_1 + \sin t\, e_2$	*unstable*
	$\rho_1 = 2 > \rho_2 = \rho_3$	$\pm e_1$	*unstable*
	$\rho_1 > \rho_2 = \rho_3 = 2$	$\cos t\, e_2 + \sin t\, e_3$	*stable*
	$\rho_1 = 2 > \rho_2 > \rho_3$	$\pm e_1$	*unstable*
	$\rho_1 > \rho_2 = 2 > \rho_3$	$\pm e_2$	*unstable*
	$\rho_1 > \rho_2 > \rho_3 = 2$	$\pm e_3$	*stable*
$\Gamma\backslash SL(2,R)$	$\rho_3 = 2 > -2 > \rho_2 > \rho_1$	$\pm e_3$	*unstable*
	$\rho_1 = 2 > -2 > \rho_2 > \rho_3$	$\pm e_1$	*stable*
$\Gamma\backslash E(2)$	$\rho_1 = \rho_2 = \rho_3 = 0$, $\mu_1 = \mu_2 = 0, \mu_3 > 0$	$\pm e_3$, $\cos t\, e_1 + \sin t\, e_2$	*stable* *unstable*
	$\rho_1 = 2 > \rho_3 > \rho_2 = -2$	$\pm e_1$	*unstable*
$\Gamma\backslash E(1,1)$	$\rho_3 = 2 > \rho_1 = -2 > \rho_2$	$\pm e_3$	*stable*
	$\rho_1 = 2 > \rho_2 = -2 > \rho_3$	$\pm e_1$	*stable*
$\Gamma \setminus Nil^3$	$\rho_1 = 2 > \rho_2 = \rho_3 = -2$	$\pm e_1$	*stable*
T^3	$\rho_1 = \rho_2 = \rho_3 = 0$, $\mu_1 = \mu_2 = \mu_3 = 0$	$\mathcal{S}$	*stable*

where $\mathcal{S}$ *stands for arbitrary left-invariant unit vector field of the form* $\xi = \cos t \cos s\, e_1 + \cos t \sin s\, e_2 + \sin t\, e_3$ *with fixed parameters* t *and* s*.*

Proof. If one take the left-invariant variations, then (9) takes the form

$$W(h,h) = (\frac{\rho_j^2}{4} - 1)h_i^2 + (\frac{\rho_i^2}{4} - 1)h_j^2$$

end hence if

$$min(|\rho_i|, |\rho_j|) \geq \rho_m = 2 \quad (i \neq j \neq m \neq i)$$

then $\xi = e_m$ generates **stable** totally geodesic subnanifold. If $\rho_i < 2$ or $\rho_j < 2$ then choosing $h_j \neq 0$ or $h_i \neq 0$ we get $W(h,h) < 0$ which means that the submanifold $\xi(G)$ is **unstable**.

Below, we check all unimodular three-dimensional Lie groups with left-invariant metric and corresponding totally geodesic unit vector fields on left-invariant stability or instability.

- The group $SO(3)$.

1. $\lambda_1 = \lambda_2 = \lambda_3 = 2$. Here ξ is arbitrary unit left-invariant and $\xi(G)$ is **classically stable** totally geodesic submanifold in T_1G by Theorem 2.2.

2. Put $\lambda_1 = \lambda_2 = 2 + \delta, \lambda_3 = 2$. Here $\xi = e_3$. Since

$$\rho_1 = 2(1+\delta) = \rho_2 = 2(1+\delta) > \rho_3 = 2,$$

$\xi(G)$ is **left-invariant stable** totally geodesic submanifold in T_1G.

3. Put $\lambda_1 = \lambda_2 = 2+\varepsilon, \lambda_3 = 2+\varepsilon - \sqrt{\varepsilon(\varepsilon+4)} > 0$. Rotating the frame in $e_1 \wedge e_2$ plane, we can always put $\xi = e_1$.

   The connection coefficients are

$$\mu_1 = 1 + \frac{\varepsilon - \sqrt{\varepsilon(\varepsilon+4)}}{2}, \mu_2 = 1 + \frac{\varepsilon - \sqrt{\varepsilon(\varepsilon+4)}}{2}, \mu_3 = 1 + \frac{\varepsilon + \sqrt{\varepsilon(\varepsilon+4)}}{2}.$$

   The principal Ricci curvatures are

$$\rho_1 = 2, \rho_2 = 2, \rho_3 = \frac{1}{2}\Big(2+\varepsilon - \sqrt{\varepsilon(\varepsilon+4)}\Big)^2.$$

   We have $\rho_1 = \rho_2 = 2 > \rho_3 > 0$ and $W(h,h) = (\rho_3^2/4 - 1)h_2^2 < 0$ for $h_2 \neq 0$. Hence, $\xi(G)$ is **unstable** totally geodesic submanifold in T_1G.

4. Put $\lambda_1 = 2, \lambda_2 = \lambda_3 = 2-\varepsilon$, $0 < \varepsilon < 2$. Here $\xi = e_1$. The connection coefficients and the Ricci principal curvatures are

$$\mu_1 = 1-\varepsilon, \mu_2 = 1, \mu_3 = 1; \qquad \rho_1 = 2, \rho_2 = 2(1-\varepsilon), \rho_3 = 2(1-\varepsilon).$$

   We have $\rho_1 = 2 > \rho_2 = \rho_3 > -2$ and hence $\rho_2^2 = \rho_3^2 < 4$. Therefore, $\xi(G)$ is **unstable** totally geodesic submanifold in T_1G.

5. Put $\lambda_1 = \varepsilon + \sqrt{4+\varepsilon^2}, \lambda_2 = \sqrt{4+\varepsilon^2}, \lambda_3 = \sqrt{4+\varepsilon^2}$. In this case $\xi = \cos t e_2 + \sin t e_3$. Rotating the frame, we may put $\xi = e_3$. Then

$$\mu_1 = \frac{\sqrt{\varepsilon^2+4} - \varepsilon}{2}, \quad \mu_2 = \mu_3 = \frac{\sqrt{\varepsilon^2+4} + \varepsilon}{2} = 1/\mu_1,$$

   The principal Ricci curvatures are

$$\rho_1 = 2 + \varepsilon(\sqrt{\varepsilon^2+4} + \varepsilon) > \rho_2 = \rho_3 = 2$$

   and hence $\xi(G)$ is **left-invariant stable** totally geodesic submanifold in T_1G

6. $\lambda_1 > \lambda_2 > \lambda_3 > 0, \quad \lambda_m^2 - (\lambda_i - \lambda_k)^2 = 4$. Denote $\lambda_2 - \lambda_3 = \delta > 0, \lambda_1 - \lambda_2 = \varepsilon > 0$. Then $\lambda_1 - \lambda_3 = \varepsilon + \delta$. Here we have 3 distinct cases.

   (i) $\lambda_1^2 = (\lambda_2 - \lambda_3)^2 + 4$, $\xi = e_1$. Then

$$\lambda_1 = \sqrt{4+\delta^2}, \lambda_2 = \sqrt{4+\delta^2} - \varepsilon > 0, \lambda_3 = \sqrt{4+\delta^2} - \varepsilon - \delta > 0.$$

   The connection coefficients are

$$\mu_1 = \frac{\sqrt{\delta^2+4} - \delta}{2} - \varepsilon, \quad \mu_2 = \frac{\sqrt{\delta^2+4} - \delta}{2}, \quad \mu_3 = \frac{\sqrt{\delta^2+4} + \delta}{2}.$$

   The principal Ricci curvatures are

$$\rho_1 = 2 > \rho_2 = 2 - \varepsilon(\sqrt{\delta^2+4} + \delta) > \rho_3 = 2 - (\varepsilon+\delta)(\sqrt{\delta^2+4} - \delta) > -2.$$

   and we obtain **unstable** totally geodesic submanifold in in T_1G.

(ii) $\lambda_2^2 = (\lambda_1 - \lambda_3)^2 + 4$, $\xi = e_2$. Then

$$\lambda_1 = \sqrt{4 + (\varepsilon + \delta)^2} + \varepsilon, \quad \lambda_2 = \sqrt{4 + (\varepsilon + \delta)^2}, \quad \lambda_3 = \sqrt{4 + (\varepsilon + \delta)^2} - \delta > 0$$

The connection coefficients are

$$\mu_1 = \frac{\sqrt{(\varepsilon + \delta)^2 + 4} - (\varepsilon + \delta)}{2}, \quad \mu_2 = \frac{\sqrt{(\varepsilon + \delta)^2 + 4} + \varepsilon - \delta}{2},$$

$$\mu_3 = \frac{\sqrt{(\varepsilon + \delta)^2 + 4} + \varepsilon + \delta}{2}$$

The principal Ricci curvatures are

$$\rho_1 = 2 + \varepsilon(\sqrt{(\varepsilon + \delta)^2 + 4} + \varepsilon + \delta), \rho_2 = 2, \rho_3 = 2 - \delta\Big(\sqrt{(\varepsilon + \delta)^2 + 4} - (\varepsilon + \delta)\Big).$$

Here $\rho_1 > \rho_2 = 2 > \rho_3 > -2$ and we obtain **unstable** totally geodesic submanifold in in T_1G.

(iii) $\lambda_3^2 = (\lambda_1 - \lambda_2)^2 + 4$, $\xi = e_3$. Then

$$\lambda_1 = \sqrt{4 + \varepsilon^2} + \varepsilon + \delta, \quad \lambda_2 = \sqrt{4 + \varepsilon^2} + \delta, \quad \lambda_3 = \sqrt{4 + \varepsilon^2}.$$

The connection coefficients are

$$\mu_1 = \frac{\sqrt{\varepsilon^2 + 4} - \varepsilon}{2}, \quad \mu_2 = \frac{\sqrt{\varepsilon^2 + 4} + \varepsilon}{2} = 1/\mu_1, \quad \mu_3 = \frac{\sqrt{\varepsilon^2 + 4} + \varepsilon}{2} + \delta$$

The principal Ricci curvatures are

$$\rho_1 = 2 + (\varepsilon + \delta)(\sqrt{\varepsilon^2 + 4} + \varepsilon) > \rho_2 = 2 + \delta(\sqrt{\varepsilon^2 + 4} - \varepsilon) > \rho_3 = 2$$

and we obtain **left-invariant stable** totally geodesic submanifold in in T_1G.

- The group $SL(2, R)$. Here we have $\xi = e_3$ or $\xi = e_1$.

1. In the case $\xi = e_3$ we have $\lambda_3^2 - (\lambda_1 - \lambda_2)^2 = 4$. Put $\lambda_1 - \lambda_2 = \varepsilon > 0$. Then $\lambda_3 = -\sqrt{4 + \varepsilon^2}$, $\lambda_2 = a > 0, \lambda_1 = a + \varepsilon$. The connection coefficients are

$$\mu_1 = -\frac{\sqrt{\varepsilon^2 + 4} + \varepsilon}{2}, \quad \mu_2 = -\frac{\sqrt{\varepsilon^2 + 4} - \varepsilon}{2} = 1/\mu_1, \quad \mu_3 = a + \frac{\sqrt{\varepsilon^2 + 4} + \varepsilon}{2}.$$

The principal Ricci curvatures are

$$\rho_1 = -2 - a(\sqrt{\varepsilon^2 + 4} - \varepsilon), \rho_2 = -2 - (a + \varepsilon)(\sqrt{4 + \varepsilon^2} + \varepsilon), \rho_3 = 2.$$

Observe that $\rho_3 = 2 > \rho_1 > \rho_2$, but $\rho_2 < \rho_1 < -2$. Therefore, $\rho_2^2 > \rho_1^2 > 4$ and hence $(\rho_2^2/4 - 1)h_3^2 + (\rho_3^2/4 - 1)h_2^2 > 0$. So we have $\xi(G)$ **left-invariant stable** totally geodesic submanifold in in T_1G.

2. In the case $\xi = e_1$ we have $\lambda_1^2 - (\lambda_2 - \lambda_3)^2 = 4$. Put $\lambda_3 = -a$ $(a > 0)$, $\lambda_2 = \lambda_3 + \varepsilon = \varepsilon - a > 0$, $\lambda_1 = \sqrt{4 + \varepsilon^2}$. (Observe, that $\lambda_1 - \lambda_2 = \sqrt{\varepsilon^2 + 4} - \varepsilon + a > -\lambda_3 = a$). Besides, $\lambda_1 \geq \lambda_2$. Therefore $\sqrt{\varepsilon^2 + 4} \geq \varepsilon - a > 0$.

   The connection coefficients are

$$\mu_1 = -a - \frac{\sqrt{\varepsilon^2 + 4} - \varepsilon}{2}, \quad \mu_2 = \frac{\sqrt{\varepsilon^2 + 4} - \varepsilon}{2}, \quad \mu_3 = \frac{\sqrt{\varepsilon^2 + 4} + \varepsilon}{2} = 1/\mu_2.$$

   The principal Ricci curvatures are

$$\rho_1 = 2, \quad \rho_2 = -2 - a(\sqrt{\varepsilon^2 + 4} + \varepsilon), \quad \rho_3 = -2 + (\varepsilon - a)(\sqrt{\varepsilon^2 + 4} - \varepsilon).$$

   Observe, that $\rho_2 < -2$ but $-2 < \rho_3 < 2$. Indeed, $\varepsilon - a \leq \sqrt{\varepsilon^2 + 4}$ and hence

$$(\varepsilon - a)(\sqrt{\varepsilon^2 + 4} - \varepsilon) \leq \sqrt{\varepsilon^2 + 4}(\sqrt{\varepsilon^2 + 4} - \varepsilon) = 4 - \varepsilon(\sqrt{\varepsilon^2 + 4} - \varepsilon) < 4.$$

   Therefore the $\xi(G)$ is **unstable** totally geodesic submanifold in in T_1G.

• The group $E(2)$. The flat case was considered in Theorem 2.3. Consider the case $\lambda_1^2 - \lambda_2^2 = 4$, $\lambda_1 > 0, \lambda_2 > 0$ and $\xi = e_1$. Put $\lambda_1 = \sqrt{4 + a^2}$, $\lambda_2 = a > 0$, $\lambda_3 = 0$. Then

$$\mu_1 = -\frac{\sqrt{4 + a^2} - a}{2}, \quad \mu_2 = \frac{\sqrt{4 + a^2} - a}{2}, \quad \mu_3 = \frac{\sqrt{4 + a^2} + a}{2} = 1/\mu_2$$

and

$$\rho_1 = 2, \rho_2 = -2, \rho_3 = -2 + a(\sqrt{4 + a^2} - a).$$

So we have

$$\rho_1 = 2 > \rho_3 > \rho_2 = -2, \quad \rho_3^2 < 4$$

and hence $(\rho_3^2/4 - 1)h_2^2 < 0$ for $h_2 \neq 0$. Therefore, $\xi(G)$ is **unstable** totally geodesic submanifold in T_1G.

• The group $E(1,1)$. Here again we have 2 options.

1. Consider $\lambda_3^2 - \lambda_1^2 = 4$, $\lambda_1 > 0, \lambda_2 = 0, \lambda_3 < 0$. The field here is $\xi = e_3$. Put $\lambda_1 = a$, $\lambda_3 = -\sqrt{a^2 + 4}$. Then

$$\mu_1 = -\frac{a + \sqrt{a^2 + 4}}{2}, \quad \mu_3 = \frac{a + \sqrt{a^2 + 4}}{2}, \quad \mu_2 = \frac{a - \sqrt{a^2 + 4}}{2} = 1/\mu_1$$

   and the principal Ricci curvatures are

$$\rho_1 = -2, \quad \rho_3 = 2, \quad \rho_2 = -\frac{1}{2}(a + \sqrt{a^2 + 4})^2 = -2 - a(a + \sqrt{4 + a^2}).$$

   Evidently, $\rho_3 = 2 > \rho_1 = -2 > \rho_2$ but $\rho_2^2 > \rho_1^2 = 4$. Therefore,

$$(\rho_2^2/4 - 1)h_1^2 \geq 0$$

   and we have **left-invariant stable** totally geodesic submanifold in T_1G.

2. Consider $\lambda_1^2 - \lambda_3^2 = 4,\ \lambda_1 > 0, \lambda_2 = 0, \lambda_3 < 0$. In this case $\xi = e_1$. Put $\lambda_1 = \sqrt{a^2+4}$, $\lambda_3 = -a < 0$. Then

$$\mu_1 = -\frac{a+\sqrt{a^2+4}}{2}, \quad \mu_2 = \frac{\sqrt{a^2+4}-a}{2}, \quad \mu_3 = \frac{\sqrt{a^2+4}+a}{2} = 1/\mu_2$$

and the principal Ricci curvatures are

$$\rho_1 = 2, \quad \rho_2 = -\frac{1}{2}(a+\sqrt{a^2+4})^2 = -2 - a(a+\sqrt{a^2+4}), \quad \rho_3 = -2.$$

Observe, that $\rho_1 = 2 > \rho_3 = -2 > \rho_2$ but $\rho_2^2 > \rho_3^2 = 4$. Therefore,

$$(\rho_2^2/4 - 1)h_1^2 \geq 0$$

and we have **left-invariant stable** totally geodesic submanifold in T_1G.

• The group Nil^3. In this case $\lambda_1 = 2, \lambda_2 = 0, \lambda_3 = 0$ and the field $\xi = e_1$. It is easy to calculate

$$\mu_1 = -1, \quad \mu_2 = 1, \quad \mu_3 = 1 = 1/\mu_2,$$

$$\rho_1 = 2, \quad \rho_2 = -2, \quad \rho_3 = -2$$

and observe, that $\rho_1 = 2 > \rho_2 = \rho_3 = -2$. Therefore, we have $\xi(G)$ **left-invariant stable** totally geodesic submanifold in T_1G.

• The flat torus T^3 was considered in Theorem 2.3.

■

Remark 2 The results of Theorem 2.4 that concern instability correlate with instability results from [14], where the the second variation of volume was calculated with respect to the field variations and the variation field was chosen with constant variation functions, i.e. left-invariant in our terminology.

Summarizing the results of the Theorem 2.4, we can observe that $\xi(G)$ *is stable with respect to left-invariant variations totally geodesic unit vector field if and only if* ξ *is the unit eigenvector of the Ricci operator which corresponds to minimal in absolute value principal Ricci curvature* $\rho = 2$.

3 Non-unimodular groups.

If G is three-dimensional non-unimodular Lie group with the left-invariant metric, then there is the left-invariant orthonormal frame (e_1, e_2, e_3) such that

$$[e_1, e_2] = \alpha\, e_2 + \beta\, e_3, \quad [e_1, e_3] = -\beta\, e_2 + \delta\, e_3, \quad [e_2, e_3] = 0,$$

where α, β and δ ar all constant satisfying $\alpha > \delta, \alpha \geq -\delta$. Let us call this frame by *canonical* one.

The non-unimodular group is not compact and does not admit a compact factor [15]. That is why one should consider the formula (3) over each domain $F \subset G$ with compact closeur. *We say that the* $\xi(G)$ *is unstable minimal/totally geodesic unit vector* *field if there is a domain $F \subset G$ with compact closeur such that the second variation $\delta^2 Vol_\xi(F) < 0$.*

The author described the groups which admit the totally geodesic left-invariant vector fields and the field themselves [21]. Here we complement the theorem with stability property as follows.

Theorem 3.1 *Let G be three-dimensional non-unimodular Lie group with the left-invariant metric Let ξ be left-invariant unit vector field on G and (e_1, e_2, e_3) the canonical ortonormal frame of its Lie algebra. Suppose $\xi(G) \subset T_1G$ is totally geodesic. Then*

- *$\beta = \delta = 0$ and $\xi = e_3$ is a parallel unit vector field; the $\xi(G)$ is **stable** totally geodesic submanifold in T_1G;*
- *$\alpha\delta = -1, \beta = \pm 1$ and ξ is of the form*

$$\xi = \frac{\beta}{\sqrt{1+\alpha^2}}\, e_2 + \frac{\alpha}{\sqrt{1+\alpha^2}}\, e_3;$$

*the $\xi(G)$ is **unstable** totally geodesic submanifold in T_1G;.*

Proof. As it was proved in [21], if $\beta = \delta = 0$, then $\xi = e_3$ is a field of unit normals of some totally geodesic 2-foliation on G and $A_\xi = -\nabla\xi = 0$. Hence, in (8) all the terms with ξ turn into zero. The (1) implies that $\xi(G)$ is horizontal while its field of normals is vertical. Therefore, $X_2 = K(\tilde{X}) = 0$ and $Z_1 = \pi_*(\tilde{N}) = 0$. The (8) implies

$$\widetilde{Ric}(\tilde{N}) = 0.$$

Therefore, $W(h, h) \geq 0$ and hence $\xi(G)$ is stable.

Consider the case

$$\alpha\delta = -1, \beta = \pm 1, \qquad \xi = \frac{\beta}{\sqrt{1+\alpha^2}}\, e_2 + \frac{\alpha}{\sqrt{1+\alpha^2}}\, e_3.$$

Observe, that the conditions $\alpha > \delta, \alpha \geq -\delta$ and $\alpha\delta = -1$ imply $\alpha \geq 1$. For such a vector field we have

$$x_1 = 0, \quad x_2 = \frac{\beta}{\sqrt{1+\alpha^2}}, \quad x_3 = \frac{\alpha}{\sqrt{1+\alpha^2}}. \tag{10}$$

The table of covariant derivatives is

∇	e_1	e_2	e_3
e_1	0	$\beta\, e_3$	$-\beta\, e_2$
e_2	$-\alpha\, e_2$	$\alpha\, e_1$	0
e_3	$\frac{1}{\alpha} e_3$	0	$-\frac{1}{\alpha} e_1$

.

Then

$$A_\xi = \begin{pmatrix} 0 & -\alpha\, x_2 & \frac{1}{\alpha} x_3 \\ \beta\, x_3 & 0 & 0 \\ -\beta\, x_2 & 0 & 0 \end{pmatrix}, \quad A^t_\xi = \begin{pmatrix} 0 & \beta\, x_3 & -\beta\, x_2 \\ -\alpha\, x_2 & 0 & 0 \\ \frac{1}{\alpha} x_3 & 0 & 0 \end{pmatrix}$$

and

$$A^t_\xi A_\xi = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{\alpha^2}{1+\alpha^2} & \frac{-\alpha\beta}{1+\alpha^2} \\ 0 & \frac{-\alpha\beta}{1+\alpha^2} & \frac{1}{1+\alpha^2} \end{pmatrix}.$$

Therefore, the singular values of A_ξ are 0 and 1. The corresponding singular frames are

$$s_0 = \xi, \quad s_1 = e_1, \quad s_2 = \frac{-\alpha\beta}{\sqrt{1+\alpha^2}}\, e_2 + \frac{1}{\sqrt{1+\alpha^2}}\, e_3;$$

$$f_1 = A_\xi(s_1) = \frac{\beta\alpha}{\sqrt{1+\alpha^2}}\, e_2 - \frac{1}{\sqrt{1+\alpha^2}}\, e_3, \quad f_2 = A_\xi(s_2) = e_1.$$

Hence, the tangent and normal orthonormal framing of $\xi(G)$ is given by (2) as follows

$$\tilde{e}_0 = \xi^h,$$

$$\tilde{e}_1 = \frac{\xi_*(s_1)}{|\xi_*(s_1)|} = \frac{1}{\sqrt{2}} e^h_1 - \frac{1}{\sqrt{2}}\Big(\frac{\alpha\beta}{\sqrt{1+\alpha^2}}\, e_2 - \frac{1}{\sqrt{1+\alpha^2}}\, e_3\Big)^v,$$

$$\tilde{e}_2 = \frac{\xi_*(s_2)}{|\xi_*(s_2)|} = \frac{1}{\sqrt{2}}\Big(\frac{-\alpha\beta}{\sqrt{1+\alpha^2}}\, e_2 + \frac{1}{\sqrt{1+\alpha^2}}\, e_3\Big)^h - \frac{1}{\sqrt{2}} e^v_1,$$

$$\tilde{n}_1 = \frac{\nu(f_1)}{|\nu(f_1)|} = \frac{1}{\sqrt{2}}\, e^h_1 + \frac{1}{\sqrt{2}}\Big(\frac{\alpha\beta}{\sqrt{1+\alpha^2}}\, e_2 - \frac{1}{\sqrt{1+\alpha^2}}\, e_3\Big)^v,$$

$$\tilde{n}_2 = \frac{\nu(f_2)}{|\nu(f_2)|} = \frac{1}{\sqrt{2}}\Big(\frac{-\alpha\beta}{\sqrt{1+\alpha^2}}\, e_2 + \frac{1}{\sqrt{1+\alpha^2}}\, e_3\Big)^h + \frac{1}{\sqrt{2}}\, e^v_1.$$

To calculate the partial Ricci curvature for $\xi(G)$ by (8), we need the components of the Riemannian tensor of G with respect to the canonical frame [21].

	e_1	e_2	e_3
$R(e_1,e_2)\bullet$	$\alpha^2 e_2 - \beta(\alpha-\delta)e_3$	$-\alpha^2 e_1$	$\beta(\alpha-\delta)e_1$
$R(e_1,e_3)\bullet$	$-\beta(\alpha-\delta)e_2 + \delta^2 e_3$	$\beta(\alpha-\delta)e_1$	$-\delta^2 e_1$
$R(e_2,e_3)\bullet$	0	$\alpha\,\delta\, e_3$	$-\alpha\,\delta\, e_2$

.

The derivatives of the curvature tensor need routine calculations which can be conducted with MAPLE.

	$(\nabla_\bullet R)(e_1,e_2)e_1$	$(\nabla_\bullet R)(e_1,e_2)e_2$	$(\nabla_\bullet R)(e_1,e_2)e_3$
e_1	$2\beta^2(\alpha-\delta)e_2 + \beta(\alpha^2-\delta^2)e_3$	$-2\beta^2(\alpha-\delta)e_1$	$-\beta(\alpha^2-\delta^2)e_1$
e_2	0	$\beta\alpha\,(\alpha-\delta)e_3$	$-\beta\alpha\,(\alpha-\delta)e_2$
e_3	0	$\alpha\,\delta\,(\alpha-\delta)e_3$	$-\alpha\,\delta\,(\alpha-\delta)e_2$

;

	$(\nabla_\bullet R)(e_1,e_3)e_1$	$(\nabla_\bullet R)(e_1,e_3)e_2$	$(\nabla_\bullet R)(e_1,e_3)e_3$
e_1	$\beta(\alpha^2-\delta^2)e_2 - \beta^2(\alpha-\delta)e_3$	$-\beta(\alpha^2-\delta^2)e_1$	$2\beta^2(\alpha-\delta)e_1$
e_2	0	$\alpha\,\delta\,(\alpha-\delta)e_3$	$-\alpha\,\delta\,(\alpha-\delta)e_2$
e_3	0	$-\beta\delta\,(\alpha-\delta)e_3$	$\beta\delta\,(\alpha-\delta)e_2$

;

	$(\nabla_{\bullet}R)(e_2,e_3)e_1$	$(\nabla_{\bullet}R)(e_2,e_3)e_2$	$(\nabla_{\bullet}R)(e_2,e_3)e_3$
e_1	0	0	0
e_2	$\alpha\beta(\alpha-\delta)e_2+\alpha\delta(\alpha-\delta)e_3$	$-\alpha\beta(\alpha-\delta)e_1$	$-\alpha\delta(\alpha-\delta)e_1$
e_3	$\alpha\delta(\alpha-\delta)e_2-\beta\delta(\alpha-\delta)e_3$	$-\alpha\delta(\alpha-\delta)e_1$	$\beta\delta(\alpha-\delta)e_1$

.

Take now the field of normal variation $\tilde N = h_1\tilde n_1 + h_2\tilde n_2$. To calculate $K(\tilde e_1, \tilde N)$, put

$$X_1=\frac{1}{\sqrt2}e_1,\qquad X_2=-\frac{1}{\sqrt2}\Big(\frac{\alpha\beta}{\sqrt{1+\alpha^2}}\,e_2-\frac{1}{\sqrt{1+\alpha^2}}\,e_3\Big),$$

$$Y_1=\pi_*(\tilde N)=\frac{1}{\sqrt2}\Big(h_1e_1+h_2(-\frac{\alpha\beta}{\sqrt{1+\alpha^2}}e_2+\frac{1}{\sqrt{1+\alpha^2}}e_3)\Big),$$

$$Y_2=K(\tilde N)=\frac{1}{\sqrt2}\Big(h_2e_1+h_1(\frac{\alpha\beta}{\sqrt{1+\alpha^2}}e_2-\frac{1}{\sqrt{1+\alpha^2}}e_3)\Big).$$

and apply (8). The MAPLE calculations yield

$$\big\langle R(X_1,Y_1)Y_1,X_1\big\rangle=-\frac14\frac{\alpha^4+\alpha^2+1}{\alpha^2}h_2^2,\quad ||R(X_1,Y_1)\xi||^2=0,$$
$$||R(\xi,Y_2)X_1+R(\xi,X_2)Y_1||^2=\frac{\alpha^6-\alpha^4+3\alpha^2+1}{\alpha^2(1+\alpha^2)}h_2^2,$$
$$\|X_2\|^2\|Y_2\|^2-\big\langle X_2,Y_2\big\rangle^2=\tfrac14h_2^2,$$
$$\big\langle R(X_1,Y_1)Y_2,X_2\big\rangle=\frac14\frac{\alpha^4+\alpha^2+1}{\alpha^2}h_2^2,\quad \big\langle R(\xi,X_2)X_1,R(\xi,Y_2)Y_1\big\rangle=0,$$
$$\big\langle(\nabla_{X_1}R)(\xi,Y_2)Y_1,X_1\big\rangle=-\frac14\frac{\alpha^6-\alpha^4+5\alpha^2-1}{\alpha^2(1+\alpha^2)}h_2^2,$$
$$\big\langle(\nabla_{Y_1}R)(\xi,X_2)X_1,Y_1\big\rangle=-\frac14\frac{\alpha^6+10\alpha^4+4\alpha^2+7}{\alpha^2(1+\alpha^2)}h_2^2.$$

After substitution into (8) and the MAPLE algebraic transformations we get

$$\tilde K(\tilde e_1,\tilde N)=\frac14\frac{\alpha^6+10\alpha^4+4\alpha^2+7}{\alpha^2(1+\alpha^2)}h_2^2.$$

In a similar way,

$$\tilde K(\tilde e_2,\tilde N)=\frac14\frac{5\alpha^8-\alpha^6+3\alpha^4+13\alpha^2-8}{\alpha^2(1+\alpha^2)^2}h_1^2+\frac{\alpha^8+2\alpha^4+1}{\alpha^2(1+\alpha^2)^2}h_2^2,$$

$$\tilde K(\tilde e_0,\tilde N)=\frac14\frac{\alpha^4+14\alpha^2-11}{(1+\alpha^2)^2}h_1^2-\frac14\frac{3\alpha^4+2\alpha^2-9}{(1+\alpha^2)^2}h_2^2.$$

As a result, the partial Ricci curvature of $\xi(G)$ obtains the form

$$\widetilde{Ric}(\tilde N)=\frac14\frac{5\alpha^8+17\alpha^4+2\alpha^2-8}{\alpha^2(1+\alpha^2)^2}h_1^2+\frac14\frac{5\alpha^8+8\alpha^6+20\alpha^4+20\alpha^2+11}{\alpha^2(1+\alpha^2)^2}h_2^2.$$

In this case we can not consider the left-invariant variations because of the boundary conditions. Nevertheless, one can consider the left-invariant variation over a subdomain $F_1 \subset F$ such that $mes(\bar F \setminus F_1) < \varepsilon$ for however small ε. If the second left-invariant variation over F_1 is negative and bounded away from zero, then by taking F_1 sufficiently large we always can make $\delta^2 Vol_\xi(F) < 0$.

If the variation field $\tilde N$ is left-invariant, then

$$\sum_{i=0}^{2} ||\tilde\nabla^{\perp}_{\tilde e_i}\tilde N||^2 = \Big(\sum_{i=0}^{2}(\tilde\gamma^2_{1i})^2\Big)(h_1^2+h_2^2),$$

where $\tilde\gamma^2_{1i} = \tilde g(\tilde\nabla_{\tilde e_i}\tilde n_1, \tilde n_2)$ are the coefficients of the $\xi(G)$ normal bundle connection with respect to the chosen frame.

Calculating, we get

$$\tilde\nabla_{\tilde e_0}\tilde n_1 = \frac{3}{4}\frac{\sqrt2}{\sqrt{1+\alpha^2}}\Big(\beta\alpha e_2 + e_3\Big)^h + \frac{\sqrt2}{1+\alpha^2}e_1^v, \quad \tilde\nabla_{\tilde e_1}\tilde n_1 = 0,$$
$$\tilde\nabla_{\tilde e_2}\tilde n_1 = \frac{1}{2}\frac{1}{\sqrt{1+\alpha^2}}\Big(\beta(2\alpha^2-1)e_2 - \alpha e_3\Big)^h - \frac{1}{2}\frac{\alpha^4+\alpha^2-2}{\alpha(1+\alpha^2)^2}e_1^v.$$

Now one can easily find

$$\tilde\gamma^2_{10} = \frac{\alpha^2+2}{1+\alpha^2}, \quad \tilde\gamma^2_{11} = 0, \quad \tilde\gamma^2_{12} = -\frac{1}{4}\frac{\sqrt2(3\alpha^2-2)}{\alpha}.$$

After substitution and MAPLE algebraic transformations, the left-invariant part of integrand in (3) takes the form

$$W(h,h) = -\frac{1}{8}\frac{\alpha^8-14\alpha^6+13\alpha^4-24\alpha^2-20}{\alpha^2(1+\alpha^2)^2}h_1^2 - \frac{1}{8}\frac{\alpha^8+2\alpha^6+19\alpha^4+12\alpha^2+18}{\alpha^2(1+\alpha^2)^2}h_2^2.$$

The factor at h_2 is always negative and hence the submanifold $\xi(G)$ is **unstable**. The proof is complete.

■

Closing observation. Analyzing the Remarks 1 and 2 one can *conjecture* that if the horizontal and vertical projections of classical normal variation vector field are in $\xi^\perp$, then the classical stability or instability of minimal (or totally geodesic) submanifold $\xi(M) \subset T_1M$ is equivalent to stability or instability of the unit vector field in the meaning of [11].

References

[1] Besse A., Einstein Manifolds, Vol. 1, 1987.

[2] Boeckx E. and Vanhecke L., *Unit tangent sphere bundles with constant scalar curvature,* Czechoslovak Math. J. **51** (2001), 523 – 544.

[3] Boeckx E. and Vanhecke L., *Harmonic and minimal radial vector fields.* Acta Math. Hungar. **90** (2001), 317 – 331.

[4] Boeckx E. and Vanhecke L., *Harmonic and minimal vector fields on tangent and unit tangent bundles* Differential Geom. Appl. **13** (2000), 77 – 93.

[5] Boeckx E. and Vanhecke L., *Isoparametric functions and harmonic and minimal unit vector fields*, Contemp. Math. **288** (2001), 20–31.

[6] Borisenko A. and Yampolsky A. *Riemannian geometry of bundles.* Uspehi Mat. Nauk, 26/6 (1991), 51-95; Engl. transl.: Russian Math. Surveys, 46/6 (1991), 55–106.

[7] Dombrowski P., *On the geometry of the tangent bundle*, J. Reine Angew. Mathematik, **210** (1962), 73-88.

[8] Gil-Medrano O., González-Dávila J.C. and Vanhecke L., *Harmonic and minimal invariant unit vector fields on homogeneous Riemannian manifolds*, Houston J. Math. 27 (2001), 377-409.

[9] Gil-Medrano O., *Relationship between volume and energy of unit vector fields*, Diff. Geom. Appl. **15** (2001), 137 – 152.

[10] Gil-Medrano O., Llinares-Fuster E., Minimal unit vector fields, *Tôhoku Math. J.*, 54 (2002), 71 – 84.

[11] Gil-Medrano O., Llinares-Fuster E., *Second variation of volume and energy of vector fields. Stability of Hopf vector fields,* Math. Ann. 320 (2001), 531-545.

[12] Gluck H. and Ziller W., *On the volume of a unit vector field on the tree sphere*, Comment. Math. Helv. **61** (1986), 177 – 192.

[13] González-Dávila J.C. and Vanhecke L., *Examples of minimal unit vector fields*, Ann Glob. Anal. Geom. **18** (2000), 385 – 404.

[14] González-Dávila J.C. and Vanhecke L., *Energy and volume of unit vector fields on three-dimensional Riemannian manifolds*, Diff. Geom. Appl. **16** (2002), 225 – 244.

[15] Milnor J., *Curvatures of left invariant metrics on Lie groups,* Adv. in Math, 21 (1976), 293 – 329.

[16] Simons J., *Minimal Varieties in Riemannian Manifolds,* Annals of Mathematics Second Series **88/1** (Jul., 1968), 62 – 105.

[17] Tsukada K. and Vanhecke L., *Invariant minimal unit vector fields on Lie groups*, Period. Math. Hungar. **40** (2000), 123 – 133.

[18] Yampolsky A., *Totally geodesic property of Hopf vector field*, Acta Math. Hungar. **101/1-2** (2003), 93 – 112.

[19] Yampolsky A., *Full description of totally geodesic unit vector field on Riemannian 2-manifold*, Journal of Mathematical Physics, Analysys, Geometry **1/3** 2004, 355 – 365.

[20] Yampolsky A., *On special types of minimal and totally geodesic unit vector fields*, Proceedings of the Seventh International Conference in Geometry, Integrability and Quantization June 2 – 10, 2005, Varna, Bulgaria, 292 – 306.

[21] Yampolsky A., *Invariant totally geodesic unit vector fields on three-dimensional Lie groups,* Journal of Mathematical Physics, Analysys, Geometry **3/2** (2007), 253 – 276.

[22] Yampolsky A., *On stability of totally geodesic unit vector field on 2-manifolds*, 8-th Int. Conf. on Topology, Geometry and teaching. Cherkassy 2013 (to appear).

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